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RESEARCH NOTES

CONDUCTION WITH A TIME-VARYING RADIATION BOUNDARY CONDITION

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NOMENCLATURE

d, thickness of finite slab;
k, thermal conductivity;
$\mathscr{L}$, non-dimensional thickness of finite slab ($= [2\sigma\varepsilon T_i^3/k]d$);
q, net surface heat influx;
q^*, non-dimensional net heat influx ($= q/\sigma\varepsilon T_i^4$);
Q, nominal surface heat influx;
Q^*, non-dimensional nominal heat influx ($= Q/\sigma\varepsilon T_i^4$);
s, non-dimensional thermal layer thickness ($= [2\sigma\varepsilon T_i^3/k]\,\Delta$);
t, time;
T, absolute temperature;
T^*, non-dimensional temperature ($= T/T_i$);
x, distance.

Greek symbols
α, thermal diffusivity;
Δ, thermal layer thickness;
ε, emissivity of surface;
σ, Stefan–Boltzmann constant;
τ, non-dimensional time ($= [4\alpha\sigma^2\varepsilon^2 T_i^6/k^2]\,t$).

Subscripts
b, back face value;
i, initial value;
s, surface value.

INTRODUCTION

It is desirable to have a rapid and reasonably accurate method of solving transient conduction problems which involve a radiation boundary condition at the surface. The integral method of Goodman [1] appears to be well suited to this purpose. Chambré [2] and Schneider [3] have utilized the integral method in solving such problems. Gay [4] has compared some finite difference solutions with the integral method and found that the latter can give satisfactory results.

Both Chambré and Schneider considered cases where the incoming heat flux was constant with time (or zero) for which the differential equation resulting from the integral method could be solved analytically.

The purpose of this note is to demonstrate the applicability of the integral technique to cases where the heat flux to the surface varies with time. This should be of interest to, *inter alios*, engineers carrying out preliminary design studies of radiative heat shields.

ANALYSIS

We consider one-dimensional heat conduction in a semi-infinite solid with constant properties. If the time varying heat flux to the surface is $Q(t)$, the net heat flux into the solid will be given by

$$q(t) = Q(t) - \sigma\varepsilon T_s(t)^4 \tag{1}$$

We shall assume that $Q(t)$ is analytic near $t = 0$ and that dQ/dt is continuous in the time range of interest. We now assume that the perturbation from initially uniform temperature is restricted to a depth of penetration Δ. If the temperature is assumed to drop from T_s to T_i quadratically in x/Δ, it may be shown by standard methods [1] that the resulting temperature distribution which satisfies the boundary conditions is

$$T - T_i = (T_s - T_i)\left[1 - 2\left(\frac{x}{\Delta}\right) + \left(\frac{x}{\Delta}\right)^2\right] \tag{2}$$

with

$$T_s - T_i = \frac{1}{2k}\,q(t)\,\Delta(t). \tag{3}$$

Substitution of the assumed temperature profile into the integral form of the conduction equation results in an ordinary differential equation linking Δ and q as functions of t. This equation, along with equations (1) and (3), provides the

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description of the problem. In non-dimensional form the equations are

$$\frac{d}{d\tau}(s^2 q^*) = 6q^*. \tag{4}$$

$$q^* = Q^* - T_s^{*4} \tag{5}$$

with

$$T_s^* = 1 + \tfrac{1}{4} s q^*. \tag{6}$$

When Q^* is constant, it is possible to solve this system of equations analytically to get a relationship between T_s^* and τ, as has been shown for example by Schneider [3]. This relationship is

$$\tfrac{3}{16}\tau = \int_1^{T_s^*} \frac{(y^4 - Q^*)(y - 1) - 2y^3(y - 1)^2}{(y^4 - Q^*)^3}\, dy. \tag{7}$$

Actually, Schneider assumed a cubic temperature profile, but this causes only a difference in the numerical coefficient of τ in equation (7). The above integral may be evaluated analytically when $Q^* = 0$, and numerically for each value of T_s^* when $Q^* \neq 0$.

When Q^* is a non-constant function of τ, the above method clearly breaks down and it becomes necessary to seek an alternative one. To avoid trial and error (or iterative) solutions it is desirable to use the differential form of equation (5), which after elimination of T_s^* by using equation (6), becomes

$$\frac{dq^*}{d\tau} = \frac{dQ^*}{d\tau} - (1 + \tfrac{1}{4} s q^*)^3 \frac{d}{d\tau}(s q^*). \tag{8}$$

It is now required to solve equations (4) and (8) with the initial conditions

$$s(0) = 0 \tag{9}$$

$$q^*(0) = Q^*(0) - 1. \tag{10}$$

Because of the non-linearity of the equations, their solution will, in general, have to be numerical. The major difficulty in such a numerical solution lies in the fact that the equations become singular as $\tau \to 0$. Thus, an analytical small-time or "starting" solution is required which would allow numerical integration to start away from the time origin.

To derive such a starting solution we assume that series solutions for s and q^* in terms of $\tau^{1/2}$ exist near $\tau = 0$. Then if the expansion of $Q^*(\tau)$ for small τ is

$$Q^* = a_0 + a_1\tau + a_2\tau^2 + \ldots \tag{11}$$

it may be readily shown, provided $Q^*(0) \neq 1$, that s and q^* are given for small τ by

$$s = b_1\tau^{1/2} + b_2\tau + b_3\tau^{3/2} + \ldots \tag{12}$$

$$q^* = c_0 + c_1\tau^{1/2} + c_2\tau + c_3\tau^{3/2} + \ldots \tag{13}$$

where

$$b_1 = \sqrt{6} \qquad c_0 = a_0 - 1$$

$$b_2 = 1 \qquad c_1 = \sqrt{6}\, c_0$$

$$b_3 = \frac{1}{8\sqrt{6}}\left(27c_0 - 12\frac{a_1}{c_0} - 16\right) \qquad c_2 = a_1 \frac{9}{4} \cdot c_0^2 + 5c_0$$

$$c_3 = \frac{1}{\sqrt{6}}\left(9a_1 - \frac{261}{8}c_0^2 - 22c_0 \frac{9}{4}\ c_0^3\right).$$

A similar procedure may also be followed for the case where $Q^*(0) = 1$. This corresponds to a situation where the initial net irradiation at the surface exactly balances the radiation out, making the initial net heat influx zero. The solutions for s and q^* for small τ are now

$$s = \tilde{b}_1\tau^{1/2} + \tilde{b}_2\tau + \tilde{b}_3\tau^{3/2} + \ldots \tag{14}$$

$$q^* = \tilde{c}_2\tau + \tilde{c}_3\tau^{3/2} + \tilde{c}_4\tau^2 + \ldots \tag{15}$$

where

$$\tilde{b}_1 = \sqrt{3} \qquad \tilde{c}_2 = a_1$$

$$\tilde{b}_2 = 3/10 \qquad \tilde{c}_3 = \sqrt{3}a_1$$

$$\tilde{b}_3 = \sqrt{3}\left(\frac{33}{200} + \frac{a_2}{6a_1}\right) \qquad \tilde{c}_4 = \frac{27}{10}a_1 + a_2.$$

It should be noted that both s and q^* now grow more slowly than in the previous case. Also there is no longer a singularity in $dq^*/d\tau$ (and $dT_s^*/d\tau$) at $\tau = 0$. In both the cases above, the small-time solution for T_s^* may easily be obtained from equation (6).

As an example of the first case [$Q^*(0) \neq 1$] we have solved a problem considered by Schneider, viz. a semi-infinite solid with $Q^{*1/4} = 0{\cdot}05$. The method used is of course valid for a time varying Q^*; a constant Q^* was chosen merely to enable a check against an analytical solution. The starting solutions given by equations (12) and (13) were used to obtain values of q^* and s for a small value of τ, and thereafter equations (4) and (8) were numerically integrated on a digital computer. Figure 1 shows that the values of T_s^* thus obtained check very well against those obtained by Schneider from equation (7). Note that in this case Q^* enters the numerical computation only through the starting solution, since $dQ^*/d\tau = 0$ in equation (8). Thus this example serves as a rather stiff test of the starting solution.

As an example of a problem of the second type, viz. $Q^*(0) = 1$ or $q^*(0) = 0$, we consider the case of a semi-infinite solid, initially at uniform temperature T_i, which exchanges heat with a source whose temperature, starting with T_i, increases to $5T_i$ by the time $\tau = 50$, staying constant thereafter.

For this case Q^* is

$$Q^* = 1{\cdot}0 + 24{\cdot}96\tau - 0{\cdot}2496\tau^2 \quad \tau \leqslant 50$$
$$= 625{\cdot}0 \quad \tau > 50. \tag{16}$$

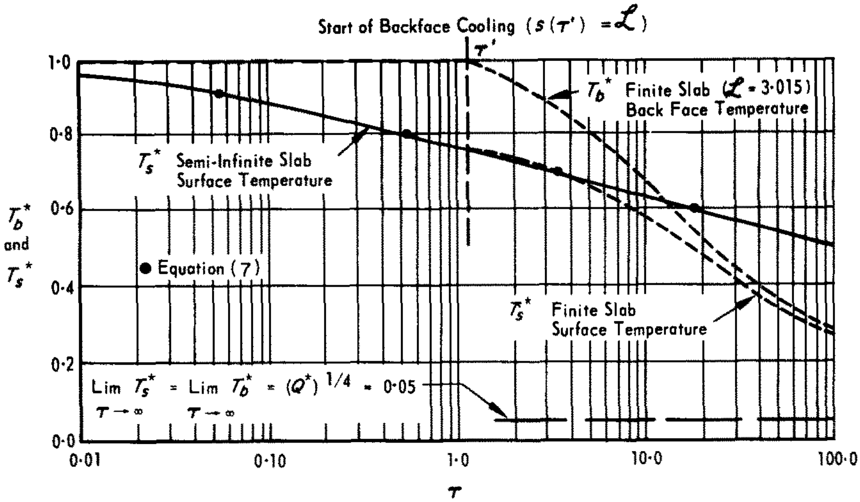

FIG. 1. Temperature variation in finite and semi-infinite slab.

We have again numerically integrated the differential equations starting from a small value of $\tau > 0$, after using the appropriate starting solutions. Figure 2 shows the resulting variation of T_s^*, q^* and s with τ. The surface temperature seems to follow the source temperature (not shown) very closely, this is consistent with the smallness of q^*.

We have so far considered only semi-infinite solids. However, the extension to finite slabs is quite straightforward, and we shall consider here a slab with an insulated back face. As shown by Goodman [1] the solution for this finite slab will be identical to that for a semi-infinite one until s is equal to the non-dimensional slab thickness, $\mathscr{L}$. After which the back temperature T_b^* will start to increase, according to the governing equations

$$12\mathscr{L}\frac{dT_b^*}{d\tau} + \mathscr{L}^2\frac{dq^*}{d\tau} = 0 \qquad (17)$$

$$4\left(T_b^* + \frac{\mathscr{L}}{4}q^*\right)^3\frac{dT_b^*}{d\tau} + \left\{1 + \left[\mathscr{L}\left(T_b^* + \frac{\mathscr{L}}{4}q^*\right)^3\right]\right\}\frac{dq^*}{d\tau} = 0 \qquad (18)$$

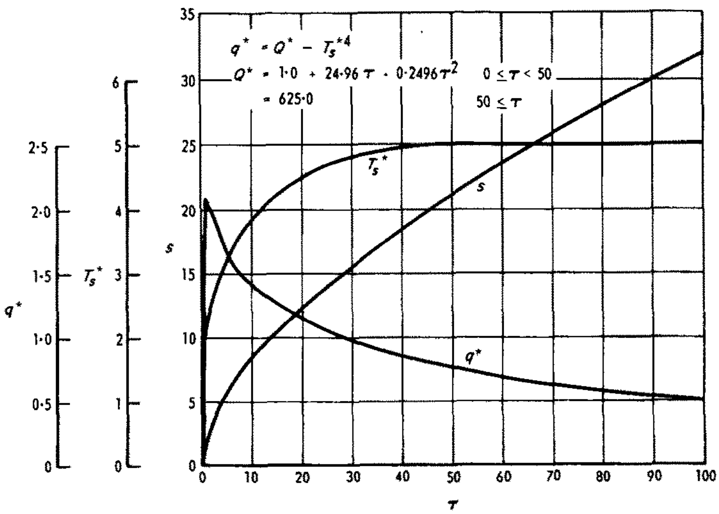

FIG. 2. Surface temperature, net heat flux and thermal layer thickness for time varying nominal heat flux.

It is again convenient to use the differential form of (5), which after elimination of T_s^*, becomes equation (18). The initial condition on q^* is obtained from the earlier solution, while initially, $T_b^* = 1$.

Schneider [3] has shown that for $Q^* = 0$, the equations for a finite slab have an analytical solution. However, for the case of constant $Q^*(\neq 0)$ he considered only the semi-infinite case. As an example of a finite slab we have considered his heat input ($Q^{*1/4} = 0{\cdot}05$) with an arbitrary but computationally convenient value of $\mathscr{L} = 3{\cdot}015$. Of course the method of solution, which was again numerical integration, is valid for non-constant Q^* also. The resulting solutions for T_b^* and T_s^* are shown in Fig. 1. The initial discontinuity in the slopes which occurs at $\tau \approx 1{\cdot}15$ (which is when $s = \mathscr{L}$) is a consequence of approximations inherent in the integral method.

CLOSURE

We have considered herein only problems involving a radiation boundary condition at the surface. However, it should be evident that similar starting solutions may be derived for fairly general forms of $q^*(T_s^*, \tau)$ provided that q^* is analytic in its arguments near $\tau = 0$.

REFERENCES

1. T. R. GOODMAN, Application of integral methods to transient non-linear heat transfer, *Advances in Heat Transfer*, edited by T. F. IRVINE, Jr. and J. P. HARTNETT, Vol. I, pp. 52–120, Academic Press, New York (1964).
2. P. L. CHAMBRÉ, Nonlinear heat transfer problem, *J. Appl. Phys.* **30**, 1683 (1959).
3. P. J. SCHNEIDER, Radiation cooling of finite heat-conducting solids, *J. Aero-Space Sci.* **27**, 547 (1960).
4. B. GAY, Comparison of methods for solution of the heat conduction equation with a radiation boundary condition, *Int. J. Heat Mass Transfer*, **8**, 507 (1965).

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THE ECKERT REFERENCE FORMULATION APPLIED TO HIGH-SPEED LAMINAR BOUNDARY LAYERS OF NITROGEN AND CARBON DIOXIDE

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NOMENCLATURE

C_f, $\tau_w/\frac{1}{2}\rho u_s^2$ = local skin friction coefficient;
C_p, specific heat at constant pressure;
h, $q_w/(T_r - T_w)$ = local heat-transfer coefficient;
h_i, $q_w/(i_r - i_w)$ = local enthalpy difference heat-transfer coefficient;
i, enthalpy;
Pr, Prandtl number;
q, heat flow per unit time and area;
r, $(T_r - T_s)/(u_s^2/2C_p)$ = recovery factor;
r_i, $(i_r - i_s)/(u_s^2/2)$ = enthalpy recovery factor;
St_i, $h_i/\rho u_s$ = Stanton number based on enthalpy difference;
T, temperature;
u, velocity.

Greek symbols
μ, viscosity;
ρ, density;
τ, shearing stress.

Subscripts
i, based on enthalpy;
r, recovery;
s, in free stream;
w, at wall.

Superscripts
$*$ reference condition.

DESIGN calculations aimed at the determination of heat